<u>Quiz 1</u>

1) (a) Express the complex number $c = (1+i)^3$ in both Cartesian and polar forms.

Step 1: expand $c = (1+i)^3$

$$c = (1+i)(1+i)(1+i) = (1 + i + i + i^2)(1+i)$$
$$= (1 + 2i + (-1))(1+i)$$
$$= 2i(1+i)$$
$$= 2i + 2i^2$$
$$= 2i + 2(-1)$$
$$= 2i - 2$$
$$\boxed{c = -2 + 2i}$$

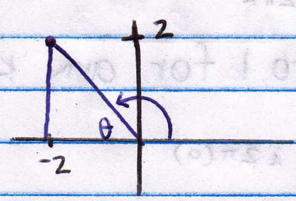
Step 2: rewrite in Cartesian form

$c = (-2, 2)$ in the complex plane.

Step 3: rewrite in polar form:

We need to rewrite c in the following form:

$z = re^{i\theta}$ where the complex number z in this case is c.



(i) We need to find θ by solving

$$\theta = \arctan\left(\frac{y}{x}\right)$$

where $y = 2$ and $x = -2$

$$\theta = \arctan\left(\frac{2}{-2}\right) = \arctan(-1) = \frac{3\pi}{4}$$

(ii) We find r by solving $r = \sqrt{x^2+y^2}$:

$r = \sqrt{(-2)^2 + (2)} = \sqrt{4+4} = \sqrt{8}$ which can also be written as $2\sqrt{2}$.

(iii) Therefore, in polar form, we write

$$c = re^{i\theta}$$

$$\boxed{c = \sqrt{8}\, e^{i3\pi/4}}$$

(b) Using polar coordinates, find all cube roots of c. (Hint: A cube root is a complex number z such that $z^3 = c$. How many complex solutions does this equation have?)

If we rewrite $z^3 = c$ as $z^3 - c = 0$, we see that the equation will have exactly three solutions (By the Fundamental Theorem of Algebra). If we write c in polar form (as in part(a)) we have $c = \sqrt{8}\, e^{i3\pi/4}$.

Therefore, $z^3 = r^3 e^{i3\theta} = \sqrt{8}\, e^{i3\pi/4}$

Since $r^3 = \sqrt{8} = \sqrt{2^3}$, And $3\theta = 3\pi/4$

$r = \sqrt{2}$ thus, $\theta = \pi/4$

Our first solution, $\boxed{z_0 = \sqrt{2}\, e^{i\pi/4}}$

To find the other solutions,

we solve $z^3 = \sqrt{2^3}\, e^{i3\pi/4} \cdot e^{i2\pi k}$

where $e^{i2\pi k}$ is equal to 1 for any $k \in \mathbb{Z}$

For $k = 0$, we have $z^3 = \sqrt{2^3}\, e^{i3\pi/4} \cdot e^{i2\pi(0)}$

which gives us $\theta_0 = \pi/4$ as our previous solution.

For $k = 1$, $z^3 = \sqrt{2^3}\, e^{i3\pi/4} \cdot e^{i2\pi}$

$= \sqrt{2^3}\, e^{i(3\pi/4 + 2\pi)} = \sqrt{2^3}\, e^{i(11\pi/4)}$

Again, $r^3 = \sqrt{8} = \sqrt{2^3}$ And $3\theta = 11\pi/4$

$r = \sqrt{2}$ $\theta = 11\pi/12$

Our second solution, $\boxed{z_1 = \sqrt{2}\, e^{i\,11\pi/12}}$

For $K=2$,

we have, $z^3 = \sqrt{2}^3 \cdot e^{i3\pi/4} \cdot e^{i2\pi(2)}$

$$= \sqrt{2}^3 \cdot e^{i(3\pi/4 + 4\pi)}$$

$$= \sqrt{2}^3 \cdot e^{i(19\pi/4)}$$

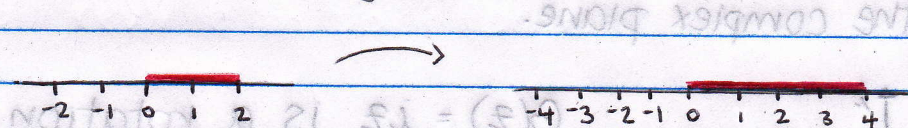
Again, $r^3 = \sqrt{2}^3$ And $3\theta = 19\pi/4$

$r = \sqrt{2}$ $\qquad \theta = 19\pi/12$

The third solution is $\boxed{z_2 = \sqrt{2}\, e^{i\,19\pi/12}}$

2) (a) Let $F(z) = 2z$. What kind of transformation is this? Briefly describe its action on the complex plane.

This is a homothetic transformation. $F(z) = 2z$ is in the form $F(z) = kz$ which means z is being stretched by a factor of 2.



Here, we can see that the plane is being stretched outward from the origin by a factor of 2.

(b) Find all fixed points of the function $F(z) = 2z$

* A fixed point for a transformation F is a solution to the equation $F(z) = z$.

Since $F(z) = 2z$, we have $2z = z$
but we cannot divide by z because it is unknown and might be 0. Thus $2z - z = 0$

$$z(2-1) = 0$$

$$z(1) = 0 \longrightarrow \boxed{z = 0}$$

(c) The transformation $F(z) = 2z$ is not an isometry of the plane. Prove this.

We need to prove that $|F(z_1) - F(z_2)| = |z_1 - z_2|$ when trying to show that a transformation is an isometry.

For this specific case,

$$d(F(z_1), F(z_2)) = |2z_1 - 2z_2|$$
$$= 2 \cdot |z_1 - z_2|$$

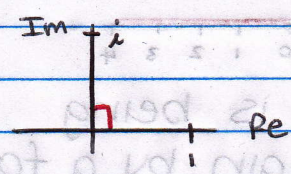
Let's take $z_1 = 1$ and $z_2 = 0$,

$$2|z_1 - z_2| = 2|1 - 0| = 2$$

which is obviously different than

$$|z_1 - z_2| = |1 - 0| = 1.$$

3) (a) Let $G(z) = iz$. What kind of transformation is this? Briefly describe its action on the complex plane.



$G(z) = iz$ is a rotation by the angle $90°$ about the origin.

Since rotations are written as $w = e^{i\theta} z$ where the real number θ is the angle of rotation,

This transformation is written as:

$$\boxed{T(x) = e^{i\pi/2} z}$$

(b) Find all fixed points of the function $G(z) = iz$

We need to solve $G(z) = z$.
Given $G(z) = iz$, we have: $iz = z$
Again, we can't divide by z because it is unknown.
Thus, $iz = z \Rightarrow iz - z = 0$

$$z(i-1) = 0$$
$$z = \frac{0}{i-1}$$
$$\boxed{z = 0}$$

(c) The transformation $G(z) = iz$ is an isometry of the plane. Prove this.

Let $z_1, z_2 \in \mathbb{C}$.

$$\begin{aligned} d(G(z_1), G(z_2)) &= |G(z_1) - G(z_2)| \\ &= |iz_1 - iz_2| = |i(z_1 - z_2)| \\ &= |i| \cdot |(z_1 - z_2)| \quad \text{by the Homogeneity property of the modulus.} \\ &= 1 \cdot |(z_1 - z_2)| \\ &= |z_1 - z_2| \\ &= d(z_1, z_2) \end{aligned}$$

This proves that $e^{i\pi/2}$ is just a rotation, so lengths in the plane will still be the same. Therefore, $G(z) = iz$ is an isometry.